Ulrich ideals of dimension one

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§1 Introduction

- In 1987 [Brennan-Herzog-Ulrich]
  ··· *Maximally Generated Maximal Cohen-Macaulay modules*
- In 2014 [Goto-Ozeki-Takahashi-Watanabe-Yoshida]
  ··· *Ulrich ideals and modules*
- Recently [Goto-Ozeki-Takahashi-Watanabe-Yoshida]
  ··· *Ulrich ideals/modules over two-dimensional rational singularities*

Question 1.1

How many Ulrich ideals are contained in a given Cohen-Macaulay local ring of dimension one?

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Contents

Notation

In what follows, unless other specified, we assume

1. $(R, \mathfrak{m})$ a Cohen-Macaulay local ring, $\dim R = 1$
2. I an $\mathfrak{m}$-primary ideal of R, $n = \mu_R(I)$
3. I contains a parameter ideal $Q = (a)$ of R as a reduction
4. $\mathrm{e}(R)$ the multiplicity of R

§2 Survey on Ulrich ideals

Based on the paper
[Goto-Ozeki-Takahashi-Watanabe-Yoshida, 2014] *Ulrich ideals and modules*

Definition 2.1

We say that I is *an Ulrich ideal of R*, if

(1) $I \supsetneq Q$, $I^2 = QI$, and

(2) I/I^2 is R/I-free.

Notice that

- (1) $\Longleftrightarrow$ $\mathrm{gr}_I(R)$ is Cohen-Macaulay ring with $\mathrm{a}(\mathrm{gr}_I(R)) = 0$.
- Suppose that $I = \mathfrak{m}$. Then
  (1) $\Longleftrightarrow$ R is not a RLR, $\mu_R(\mathfrak{m}) = \mathrm{e}(R)$.

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- $(1) \iff \mathrm{gr}_I(R)$ is Cohen-Macaulay ring with $\mathrm{a}(\mathrm{gr}_I(R)) = 0$.
- Suppose that $I = \mathfrak{m}$. Then
  $(1) \iff R$ is not a RLR, $\mu_R(\mathfrak{m}) = \mathrm{e}(R)$.

Example 2.2

Let A be a Cohen-Macaulay local ring with $\dim R = 1$, F a finitely generated free A-module. Let

$$R = A \ltimes F, \quad (a, x)(b, y) := (ab, ay + bx)$$

be the idealization of F over A. We put

$$I = \mathfrak{p} \times F, \quad Q = \mathfrak{p}R,$$

where $\mathfrak{p}$ is a parameter ideal of A. Then I is an Ulrich ideal of R with $\mu_R(I) = \operatorname{rank}_A F + 1$.

Let $\mathcal{X}_R$ be the set of Ulrich ideals of R.

Theorem 2.3

Suppose that R is of finite CM-representation type. Then $\mathcal{X}_R$ is a finite set.

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Let

$$R = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]] \subseteq V = k[[t]]$$

be the numerical semigroup ring over a field k, where $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ such that $\gcd(a_1, a_2, \ldots, a_\ell) = 1$.
We define

$$\mathrm{o}(f) := \max\{n \in \mathbb{Z} \mid f \in t^n V\}$$

for $0 \neq f \in V$.
We set

$$\mathcal{X}_R^g = \{\text{Ulrich ideals of } R \text{ generated by monomials in } t\}.$$

Theorem 2.4

The set $\mathcal{X}_R^g$ is finite.

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We continue the researches ([GOTWY]), providing a practical method for counting Ulrich ideals in dimension one.

Lemma 2.5

Suppose that $I^2 = QI$. Then TFAE.

(1) I *is an Ulrich ideal of* R.

(2) I/Q *is a free* R/I*-module.*

Proof.

The equivalence of (1) and (2) follows from the splitting of the sequence

$$0 \to Q/QI \to I/I^2 \to I/Q \to 0.$$

When this is the case, $I/Q \cong (R/I)^{n-1}$, since $Q = (a)$ is generated by a part of a minimal basis of I. □

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Let $I \in \mathcal{X}_R$. Look at the isomorphism

$$I/Q \cong (R/I)^{n-1}.$$

Then we have the following. Here $\mathrm{r}(R) = \ell_R(\mathrm{Ext}_R^1(R/\mathfrak{m}, R))$.

Corollary 2.6

(1) $Q : I = I$.

(2) $0 < (n-1)\cdot\mathrm{r}(R/I) = \mathrm{r}_R(I/Q) \leq \mathrm{r}(R/Q) = \mathrm{r}(R)$.

Hence $n \leq \mathrm{r}(R) + 1$.

Therefore, if R is a Gorsenstein ring, then R/I is Gorenstein, $n = 2$ and I is a good ideal in the sense of [2].

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Let $I \in \mathcal{X}_R$. Let

$$\mathbb{F}_\bullet : \quad \cdots \to F_i \overset{\partial_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{\partial_1}{\to} F_0 \to R/I \to 0$$

be a minimal free resolution of R/I and $\beta_i = \operatorname{rank}_R F_i \quad (i \geq 0)$.

Theorem 2.7

(1) $R/I \otimes_R \partial_i = 0$ *for* $\forall i \geq 1$.

(2) $\beta_i = \begin{cases} (n-1)^{i-1} \cdot n & (i \geq 1), \\ 1 & (i = 0). \end{cases}$

Hence $\beta_i = \binom{1}{i} + (n-1)\beta_{i-1}$ *for* $\forall i \geq 1$.

Let $I \in \mathcal{X}_R$. Let

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Look at the exact sequence

$$0 \to Q \to I \to (R/I)^{\oplus(n-1)} \to 0.$$

Corollary 2.8

A minimal free resolution of I is obtained by those of Q and $(R/I)^{\oplus(n-1)}$.

Corollary 2.9

$\mathrm{Syz}_R^{i+1}(R/I) \cong [\mathrm{Syz}_R^i(R/I)]^{\oplus(n-1)}$ *for all $i \geq 1$. Hence*

$$\mathrm{Syz}_R^{i+1}(R/I) \cong \mathrm{Syz}_R^i(R/I)$$

for all $i \geq 1$, if R is a Gorenstein local ring.

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Theorem 2.10

Let $I, J \in \mathcal{X}_R$. Then $I = J$ if and only if

$$\mathrm{Syz}_R^i(R/I) \cong \mathrm{Syz}_R^i(R/J)$$

for some $i \geq 0$.

Example 2.11

Let $I \in \mathcal{X}_R$. Suppose that R is a Gorenstein local ring with $\dim R = 1$. Then $\mu_R(I) = 2$. We write

$$I = (a, x) \quad (x \in R)$$

where $Q = (a)$ is a reduction of I. Then $x^2 = ay$ for some $y \in I$, since $I^2 = aI$. Then

$$\mathbb{F}_\bullet : \quad \cdots \to R^2 \xrightarrow{\begin{pmatrix} -x & -y \\ a & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -x & -y \\ a & x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} a & x \end{pmatrix}} R \xrightarrow{\varepsilon} R/I \to 0.$$

§3 The Gorenstein case

In this section, we assume that R is a Gorenstein ring.

Definition 3.1 ([2])

We say that I is *a good ideal of R*, if

(1) $I^2 = QI$ and

(2) $Q : I = I$.

Notice that

- I is good $\Longleftrightarrow$ $\mathrm{gr}_I(R)$ is Gorenstein with $\mathrm{a}(\mathrm{gr}_I(R)) = 0$.

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Setting 3.2

Let $\mathcal{V}_R$ be the set of intermediate rings $R \subsetneq A \subseteq \mathrm{Q}(R)$ such that A is a finitely generated R-module and put

$$\mathcal{Y}_R = \{I \mid I \text{ is a good ideal of } R\},$$

$$\mathcal{Z}_R = \{A \in \mathcal{V}_R \mid A \text{ is a Gorenstein ring}\}.$$

Hence $\mathcal{X}_R \subseteq \mathcal{Y}_R$ and $\mathcal{Z}_R \subseteq \mathcal{V}_R$.

Lemma 3.3 (Key Lemma)

There is a well-defined bijective map

$$\varphi : \mathcal{Z}_R \to \mathcal{Y}_R, \quad A \mapsto R : A.$$

Therefore, $R : A \in \mathcal{X}_R \iff \mu_R(A) = 2$ *for* $A \in \mathcal{Z}_R$*.*

Proof.

Let $A \in \mathcal{Z}_R$ and put $J = R : A$. Then $J = bA$ for some $b \in J$, since A is a Gorenstein ring and $J \cong \mathrm{K}_A$. Let $\mathfrak{q} = bR$. Then

$$J^2 = \mathfrak{q}J \quad \text{and} \quad \mathfrak{q} : J = R : A = J,$$

so that J is a good ideal of R. If $J \in \mathcal{X}_R$, then $\mu_R(A) = \mu_R(J) = 2$. Suppose that $\mu_R(A) = 2$. Then $J/\mathfrak{q}$ is cyclic, since $\mathfrak{q}$ is a minimal reduction of J. Hence $J/\mathfrak{q} \cong R/J$, because $\mathfrak{q} : J = J$. Thus $J \in \mathcal{X}_R$. □

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Let $V = \overline{R}$ be the normalization of R.

Corollary 3.4

Suppose that V is a DVR and V is a finitely generated R-module. Then TFAE.

(1) $\mathrm{e}(R) = 2$

(2) $R : V$ *is an Ulrich ideal of* R

Proof.

Let $f \in \mathfrak{m}$ such that $fV = \mathfrak{m}V$. Then fR is a reduction of $\mathfrak{m}$. Therefore we have

$$\mathrm{e}(R) = \mathrm{e}^0_{\mathfrak{m}}(R) = \mathrm{e}^0_{fR}(R) = \mathrm{e}^0_{fR}(V) = \ell_R(V/\mathfrak{m}V) = \mu_R(V).$$

□

Example 3.5

Let $R = k[[t^3, t^4]]$. Then $\mathcal{X}_R = \{(t^4, t^6)\}$.

Proof.

Let $A \in \mathcal{Z}_R$. We may assume that $R \subsetneq A \subsetneq V = k[[t]]$. Since R is Gorenstein, $t^5 \in A$ which shows $k[[t^3, t^4, t^5]] \subsetneq A$. Since $A \neq V$, then $A \subseteq k[[t^2, t^3]]$, so that

$$k[[t^3, t^4, t^5]] \subsetneq A \subseteq k[[t^2, t^3]].$$

Thus

$$A = k[[t^2, t^3]] = R + Rt^2.$$

Therefore $R : A = R : t^2 = (t^4, t^6) \in \mathcal{X}_R$. □

Example 3.6

Let $R = k[[t^4, t^5, t^6]]$. Then $\mathcal{X}_R = \{(t^4 - ct^5, t^6) \mid c \in k\}$.

Proof.

Let $A \in \mathcal{Z}_R$ such that $R \subsetneq A \subsetneq V = k[[t]]$. Then $t^7 \in A$ and hence

$$k[[t^4, t^5, t^6, t^7]] \subsetneq A \subseteq k[[t^2, t^3]].$$

Since $k[[t^3, t^4, t^5]]$ is not Gorenstein, $A \not\subseteq k[[t^3, t^4, t^5]]$, whence

$$\exists\ \xi \in A \ \text{ such that } \ \mathrm{o}(\xi) = 2.$$

We may assume $\xi = t^2 + ct^3$ where $c \in k$. Therefore

$$A = k[[t^2, t^3]] \text{ or } R[\xi].$$

Proof. (continued)

If $A = k[[t^2, t^3]]$, then

$$\mu_R(A) = \ell_R(A/\mathfrak{m}A) = 3,$$

so that $R : A \notin \mathcal{X}_R$. Suppose that $A = R[\xi]$. Then $\mathrm{e}(A) = 2$ and therefore A is Gorenstein. Since $\mathfrak{m}A = t^4V$, we have

$$\mu_R(A) = \ell_R(V/\mathfrak{m}A) - \ell_R(V/A) = 4 - 2 = 2.$$

Hence $R : A = R : \xi = (t^4 - ct^5, t^6) \in \mathcal{X}_R$. □

Therefore

- $\mathcal{X}_R = \{(t^4 - ct^5, t^6) \mid c \in k\} \overset{1:1}{\longleftrightarrow} k$
- $\mathcal{X}_R^g = \{(t^4, t^6)\}$

Proof. (continued)

If $A = k[[t^2, t^3]]$, then

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Therefore

- $\mathcal{X}_R = \{(t^4 - ct^5, t^6) \mid c \in k\} \overset{1:1}{\longleftrightarrow} k$
- $\mathcal{X}_R^g = \{(t^4, t^6)\}$

Example 3.7

- Let $R = k[[t^3, t^5]]$. Then $\mathcal{X}_R = \emptyset$.
- Let $R = k[[t^3, t^7]]$. Then $\mathcal{X}_R = \{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}$.
- Let $R = k[[t^2, t^{2\ell+1}]]$ $(\ell > 0)$. Then $\mathcal{X}_R = \{(t^{2i}, t^{2\ell+1}) \mid 1 \leq i \leq \ell\}$.

Theorem 3.8

Let $R = k[[t^n, t^{n+1}, \dots, t^{2n-2}]]$ $(n \geq 3)$. *Then*

$$\mathcal{X}_R = \begin{cases} \{(t^4, t^6)\} & (n = 3), \\ \{(t^4 - ct^5, t^6) \mid c \in k\} & (n = 4), \\ \emptyset & (n \geq 5). \end{cases}$$

Proof of the case: $n = 2q+1 \ (q \geq 2)$

Let $I \in \mathcal{X}_R$ and $A = \frac{I}{a} \subseteq \mathrm{Q}(R)$. Then

$$t^n V \subseteq k[[t^n, t^{n+1}, \ldots, t^{2n-1}]] \subseteq A,$$

since t^{2n-1} is the generator of the socle of $\mathrm{Q}(R)/R$. Let

$$\mathcal{C} := A : V = t^c V \ (c \geq 0).$$

Then $c \leq n = 2q+1$. We put $\ell = \ell_A(V/A)$. Hence $2\ell = c$, since A is Gorenstein. Thus

$$\ell_A(V/A) \leq q.$$

Look at

$$\overline{A} := A/\mathfrak{m}A \supsetneq J := \mathfrak{m}_{\overline{A}} \supsetneq J^2 = (0).$$

Take $\xi \in \mathfrak{m}_A$ so that $J = (\overline{\xi})$. Then $\overline{\xi} \neq 0$ and $\overline{\xi}^2 = 0$ in $\overline{A}$.

Proof of the case: $n = 2q+1$ $(q \geq 2)$ (continue).

Hence

$$\xi^2 \in \mathfrak{m}A \subseteq t^n V \quad \text{and} \quad A = R + R\xi,$$

because $A/\mathfrak{m}A = k + k\overline{\xi}$. Therefore $2{\cdot}\mathrm{o}(\xi) \geq n = 2q+1$, so that

$$\mathrm{o}(\xi) \geq q+1.$$

Thus

$$A = R + R\xi \subseteq T := k[[t^{q+1}, t^{q+2}, \ldots, t^{2q+1}]] \subseteq V.$$

Hence $A = T$, because

$$\ell_R(V/T) = q \quad \text{and} \quad \ell_A(V/A) \leq q.$$

This is impossible. Thus $\mathcal{X}_R = \emptyset$. □

§4 Finite Cohen-Macaulay representation type

Let $(R, \mathfrak{m})$ be a one-dimensional Gorenstein complete equi-characteristic local ring with algebraically closed residue class field $k = R/\mathfrak{m}$.

Suppose that R has finite CM-representation type. Then R is *a simple singularity*, i.e.,

$$R = k[[X, Y]]/(f),$$

where f is one of the polynomials as follows.

(A_n) $X^2 - Y^{n+1}$ $(n \geq 1)$

(D_n) $X^2Y - Y^{n-1}$ $(n \geq 4)$

(E_6) $X^3 - Y^4$

(E_7) $X^3 - XY^3$

(E_8) $X^3 - Y^5$

Type (A_n) : $X^2 - Y^{n+1}$ $(n \geq 1)$

Theorem 4.1

(1) $n = 2\ell - 1$ $(\ell \geq 1, \mathrm{ch}\, k \neq 2) \cdots$ $\mathcal{X}_R = \{(x, y^i) \mid 1 \leq i \leq \ell\}$.

(2) $n = 2\ell$ $(\ell \geq 1) \cdots$ $\mathcal{X}_R = \{(x, y^i) \mid 1 \leq i \leq \ell\}$.

Proof of Theorem 4.1 (1).

Notice that

$$\mathcal{Z}_R = \{R[\frac{x}{y^i}] \mid 1 \leq i \leq \ell\} \text{ and } \mu_R(A) = 2 \text{ for } \forall A \in \mathcal{Z}_R.$$

Therefore $R : R[\frac{x}{y^i}] = R : \frac{x}{y^i} = (x, y^i) \in \mathcal{X}_R$. □

Type (A_n) : $X^2 - Y^{n+1}$ $(n \geq 1)$

Theorem 4.1

(1) $n = 2\ell - 1$ $(\ell \geq 1, \mathrm{ch}\, k \neq 2)$ $\cdots$ $\mathcal{X}_R = \{(x, y^i) \mid 1 \leq i \leq \ell\}$.

(2) $n = 2\ell$ $(\ell \geq 1)$ $\cdots$ $\mathcal{X}_R = \{(x, y^i) \mid 1 \leq i \leq \ell\}$.

Proof of Theorem 4.1 (1).

Notice that

$$\mathcal{Z}_R = \{R[\frac{x}{y^i}] \mid 1 \leq i \leq \ell\} \text{ and } \mu_R(A) = 2 \text{ for } \forall A \in \mathcal{Z}_R.$$

Therefore $R : R[\frac{x}{y^i}] = R : \frac{x}{y^i} = (x, y^i) \in \mathcal{X}_R$. □

Type (A_n) : $X^2 - Y^{n+1}$ $(n \geq 1)$

Proof of Theorem 4.1 (2).

In this case $(n = 2\ell,\ \ell \geq 1)$,

$$R = k[[X, Y]]/(X^2 - Y^{2\ell+1}) \cong k[[t^2, t^{2\ell+1}]].$$

Since

$$\mathcal{X}_{k[[t^2, t^{2\ell+1}]]} = \{(t^{2i}, t^{2\ell+1}) \mid 1 \leq i \leq \ell\},$$

we have

$$\mathcal{X}_R = \{(x, y^i) \mid 1 \leq i \leq \ell\}.$$

□

Type $(\mathrm{E}_6) : X^3 - Y^4$, $(\mathrm{E}_7) : X^3 - XY^3$, $(\mathrm{E}_8) : X^3 - Y^5$

Theorem 4.2

- (E_6) $\mathcal{X}_R = \{(x, y^2)\}$.
- (E_7) $\mathcal{X}_R = \{(x, y^3)\}$.
- (E_8) $\mathcal{X}_R = \emptyset$.

Proof of the cases (E_6), (E_8).

- (E_6) $\cdots$ $R = k[[X, Y]]/(X^3 - Y^4) \cong k[[t^3, t^4]]$.
- (E_8) $\cdots$ $R = k[[X, Y]]/(X^3 - Y^5) \cong k[[t^3, t^5]]$.

Remember that $\mathcal{X}_{k[[t^3,t^4]]} = \{(t^4, t^6)\}$ and $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$. □

Type $(\mathrm{E}_6) : X^3 - Y^4$, $(\mathrm{E}_7) : X^3 - XY^3$, $(\mathrm{E}_8) : X^3 - Y^5$

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Proof of the cases (E_6), (E_8).

- $(\mathrm{E}_6) \cdots$ $R = k[[X,Y]]/(X^3 - Y^4) \cong k[[t^3, t^4]]$.
- $(\mathrm{E}_8) \cdots$ $R = k[[X,Y]]/(X^3 - Y^5) \cong k[[t^3, t^5]]$.

Remember that $\mathcal{X}_{k[[t^3,t^4]]} = \{(t^4, t^6)\}$ and $\mathcal{X}_{k[[t^3,t^5]]} = \emptyset$. $\square$

Proof of the case (E_7) : $X^3 - XY^3$

Due to [Goto-Takahashi-T, 2015].

Claim

$$\mathcal{Z}_R = \{k[[Y]] \oplus k[[t^2, t^3]],\ k[[Y]] \oplus k[[t]],\ k + \mathrm{J}(\overline{R})\}$$

Sketch of proof.

Let $f = X^2 - Y^3$. Let $\varphi : S = k[[X, Y]] \longrightarrow V = k[[t]]$ be the k-algebra map such that

$$\varphi(X) = t^3, \quad \varphi(Y) = t^2.$$

Then $S/(f) \cong k[[t^2, t^3]]$ and we get the following diagram.

Proof of the case (E_7) : $X^3 - XY^3$

$$0 \longrightarrow S/(X\cdot f) \xrightarrow{\alpha} S/(X) \oplus S/(f) \xrightarrow{\beta} S/(X, f) \longrightarrow 0$$

$$\begin{array}{ccccc} & & S/(X) \oplus S/(f) & & \\ & & \downarrow \wr\Vert & & \\ & \searrow j & k[[Y]] \oplus k[[t^2, t^3]] & & \\ & & \cap\vert & & \\ & & k[[Y]] \oplus V = \overline{R} & \xrightarrow{p_2} & V \end{array}$$

Let $A \in \mathcal{Z}_R$. Consider $p_2 : \overline{R} \to V,\ (a, b) \mapsto b$. We put $B = p_2(A)$. Since $k[[t^2, t^3]] \subseteq B \subseteq V$, we get

$$B = k[[t^2, t^3]] \ \text{ or } \ V.$$

Proof of the case $(\mathrm{E}_7) : X^3 - XY^3$

Case 1 (A is not a local ring.)

$$A = k[[Y]] \oplus B.$$

Case 2 (A is a local ring.)

- $B = V \cdots$ $A \cong k[[Y, Z]]/(Z(Y - Z^2)) = k + \mathrm{J}(\overline{R})$
- $B = k[[t^2, t^3]] \cdots$ A is not a Gorenstein ring.

Hence

$$\mathcal{Z}_R = \{k[[Y]] \oplus k[[t^2, t^3]], k[[Y]] \oplus V, k + \mathrm{J}(\overline{R})\}.$$

Proof of the case $(\mathrm{E}_7) : X^3 - XY^3$

Let $A \in \mathcal{Z}_R$ such that $\mu_R(A) = 2$. Then

$$A = k[[Y]] \oplus k[[t^2, t^3]],$$

so that $\mathcal{X}_R = \{R : A\}$. Since

$$0 \to R \to S/(X) \oplus S/(f) \ (= A) \to S/(X, Y^3) \to 0,$$

we have $A/R \cong S/(X, Y^3)$. Thus $R : A = (x, y^3)$. $\square$

Type (D_n) : $X^2Y - Y^{n-1}$ $(n \geq 4)$

Theorem 4.3

(1) $n = 2\ell + 1$ $(\ell \geq 2)$
$\cdots$ $\mathcal{X}_R = \{(x^2, y), (x, y^{2\ell-1})\}$.

(2) $n = 2\ell$ $(\ell \geq 2, \mathrm{ch}\, k \neq 2)$
$\cdots$ $\mathcal{X}_R = \{(x^2, y), (x - y^{\ell-1}, y(x + y^{\ell-1})), (x + y^{\ell-1}, y(x - y^{\ell-1}))\}$.

Question 4.4

Is there any relation between Ulrich ideals and representation theory?

§5 The non-Gorenstein case

Theorem 5.1

Let $(V, \mathfrak{n})$ *be a Cohen-Macaulay local ring with* $\dim V = 1$. *Let*

$$R = V[Y]/(Y^n) \quad (n \geq 2).$$

Then $\sharp \mathcal{X}_R = \infty$.

Proof of Theorem 5.1.

Suppose $n = 2q + 1$ $(q \geq 1)$. Let a be a parameter for V, and

$$I = I_\ell := (a^{2\ell} - y, a^\ell y^q) \quad \text{for} \quad \forall \ell > 0,$$

where y is the image of Y in R. Then

$$I^2 = (a^{2\ell} - y)I,$$

while $R/(a^{2\ell} - y) \cong V/(a^{2\ell n})$ and $R/I \cong V/(a^{\ell n})$. Hence

$$\ell_R(I/(a^{2\ell} - y)) = \ell_R(R/I) = \ell{\cdot}n{\cdot}\mathrm{e}(V).$$

Therefore $I/(a^{2\ell} - y) \cong R/I$, so that $I_\ell = I \in \mathcal{X}_R$. Thus $\sharp\mathcal{X}_R = \infty$.

For the case $n = 2q$ $(q \geq 1)$, consider $I = I_\ell := (a^\ell, y^q)$.

$\square$

Theorem 5.2

Suppose that $R = \widehat{R}$ and R is a reduced ring. If

$$\mathfrak{m}\overline{R} \subseteq R \quad \text{and} \quad R \neq \text{a RLR},$$

then $\mathcal{X}_R = \{\mathfrak{m}\}$.

Proof.

The ring $\overline{R}$ is a finitely generated R-module and $\mathfrak{m}\overline{R} = \mathfrak{m}$. Take $a \in \mathfrak{m}$ so that $\mathfrak{m} = a\overline{R}$. Then $\mathfrak{m}^2 = a\mathfrak{m}$ and $\mu_R(\mathfrak{m}) > 1$. Thus $\mathfrak{m} \in \mathcal{X}_R$. Conversely, let $I \in \mathcal{X}_R$ and choose a reduction $Q = (a)$ of I. Then $\mathfrak{m}\frac{I}{a} \subseteq R$, since $\frac{I}{a} \subseteq \overline{R}$. Hence $\mathfrak{m}I \subseteq Q$. Therefore $I = \mathfrak{m}$, since I/Q is R/I-free. Thus $\mathcal{X}_R = \{\mathfrak{m}\}$. □

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Corollary 5.3

Let $n \geq 2$ *and* $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$. *Then* $\mathcal{X}_R = \{\mathfrak{m}\}$.

Corollary 5.4

Let $(S, \mathfrak{n})$ *be a RLR with* $\dim S = n \geq 2$. *Let* $\mathfrak{n} = (X_1, X_2, \ldots, X_n)$ *and put*

$$R = S/\cap_{i=1}^{n}(X_j \mid j \neq i).$$

Then $\mathcal{X}_R = \{\mathfrak{m}\}$.

Corollary 5.5

Let K/k $(K \neq k)$ *be a finite extension of fields. We put*

$$V = K[[t]] \ \text{ and } \ R = k[[tK]].$$

Then $\mathcal{X}_R = \{tV\}$.

§6 Value semigroups

Let $V = k[[t]]$.

Example 6.1

(1) Let $f, g \in V$ such that $\mathrm{o}(f) = 3, \mathrm{o}(g) = 4$. We put $R = k[[f, g]]$. Then $\mathcal{X}_R = \{(g, f^2)\}$.

(2) Let $f, g \in V$ such that $\mathrm{o}(f) = 3, \mathrm{o}(g) = 5$. We put $R = k[[f, g]]$. Then $\mathcal{X}_R = \emptyset$.

(3) Let $R = k[[f_5, f_6, f_7, f_8]]$, where $f_i \in V$ such that $\mathrm{o}(f_i) = i$ for $5 \leq \forall i \leq 8$. Then $\mathcal{X}_R = \emptyset$.

Thank you very much for your attention.

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